UNIVERSITY OF PARMA - Italy

DEPT. of ENGINEERING & ARCHITECTURE

## PhD course on

# INTRODUCTION TO NON-LINEAR

# PROBLEMS IN MECHANICS

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# 3. SOLUTION OF NON-LINEAR PROBLEMS, ITERATIVE METHODS, CONVERGENCE CRITERIA

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# Summary of chapt. 3

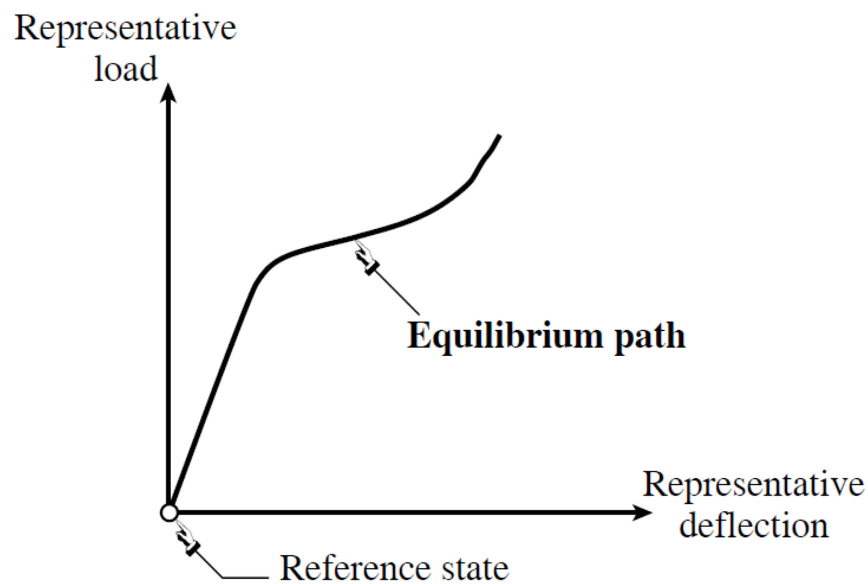
# 3.1. SOME ASPECTS OF NON LINEAR PROBLEMS

## 3.1.1 EQUILIBRIUM PATH AND RESPONSE DIAGRAMS

The concept of **equilibrium path** plays a central role in explaining the problem of nonlinear structural analysis.

This concept lends itself to graphical representation in the form of so-called **response diagrams**.

The most used form of these pictures is the **load-deflection response** diagram. Through this representation, many key concepts can be illustrated and interpreted in physical, mathematical or computational terms.

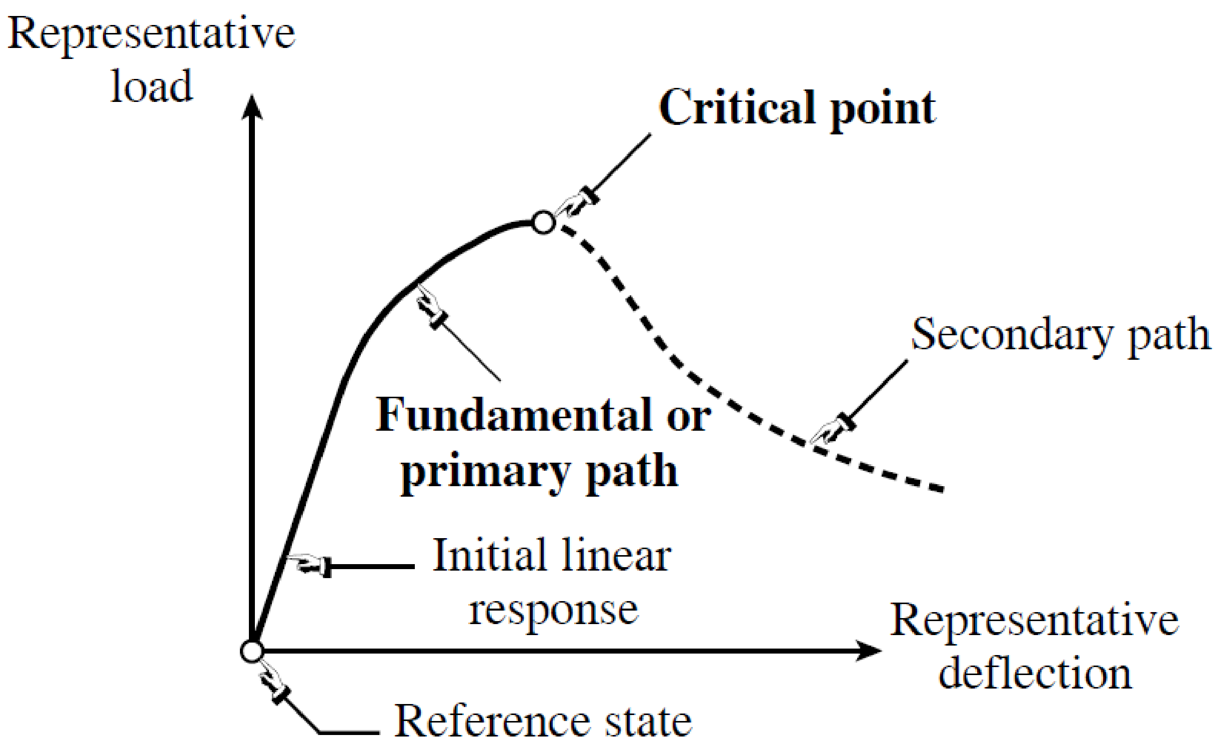


## **Load-deflection response**

The overall behavior of many structures under static loads can be characterized by a load-deflection or force-displacement response.

In this figure a "**representative**" **force** quantity is plotted against a "**representative**" **displacement** quantity.

If the response graph is nonlinear, the structure behavior is nonlinear.



The word "**representative**" implies the choice among many possibilities.

For simple structures such a choice can be trivial, for more complex structures such a decision can not be obvious.

The response curve in a load-deflection diagram is called a path: each point of such a curve represents a possible configuration or state of the mechanical system.

If the path represents configurations of static equilibrium, it is also called equilibrium path.

The origin of the response curve (zero load and displacement) is called reference state and from this configurations loads and deflections are measured.

In the case of perfect structures, the equilibrium reference state is undeformed and unstressed; the equilibrium path then passes through this reference state (origin of the response load-deflection diagram).

### 3.1.2 PARTICULAR EQUILIBIUM POINTS

Some points of the equilibrium path must be carefully considered:

***Critical points***

- Limit points: the tangent to the response curve is horizontal
- Bifurcation points: two or more equilibrium paths cross at such points

At such points the relation between load and deflection is not unique; the structure becomes physically uncontrollable.

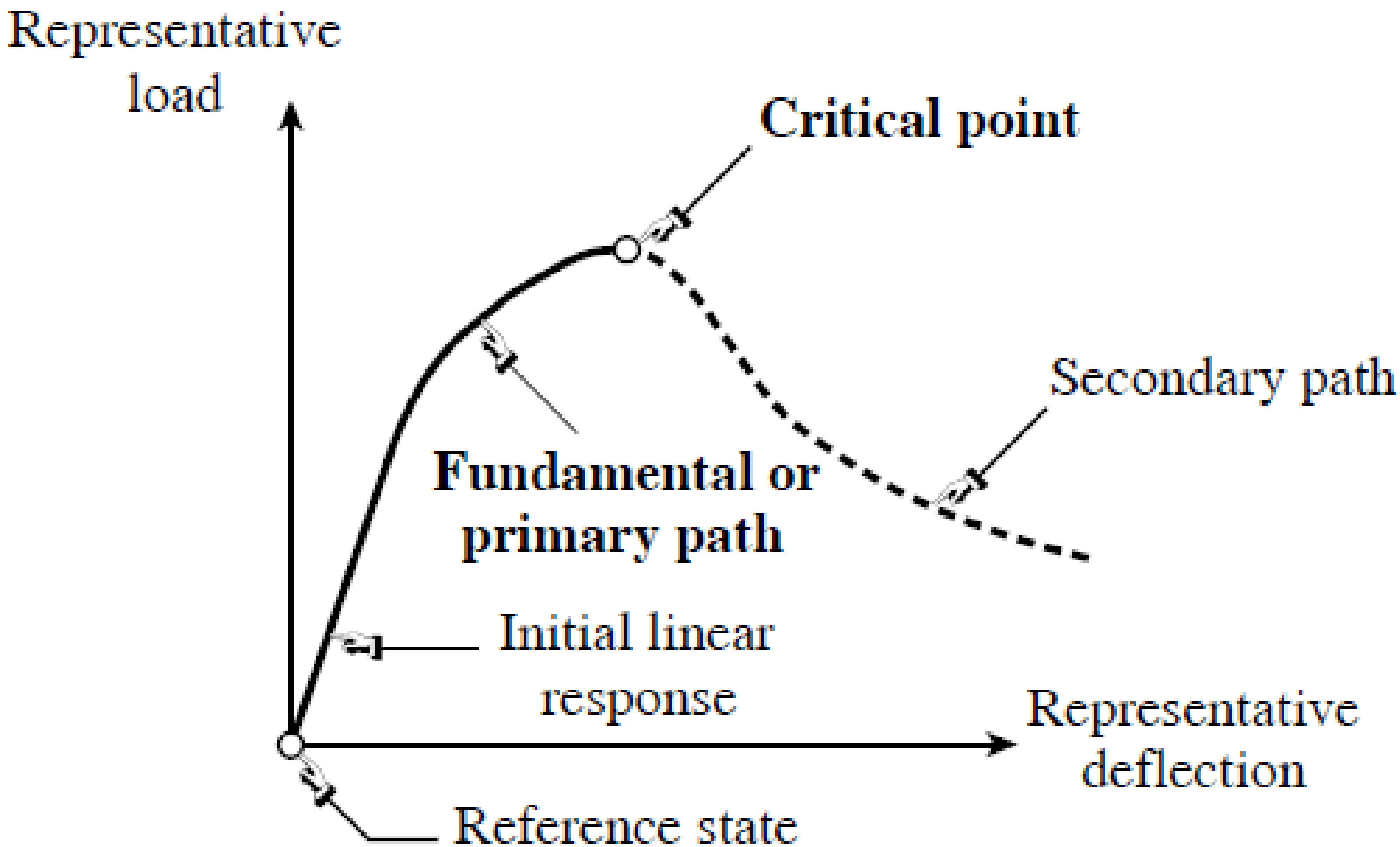
***Turning points***

- At such points the tangent to the response curve is vertical

***Failure points***

- At such points the equilibrium path breaks because of the failure of the structure; such failure can be local or global. In the first case the structure can reach a new equilibrium configuration after dynamically jumping to another equilibrium path. If the failure point correspond to a global crisis, the structure undergoes catastrophic failure.

Any path that follows a fundamental path by connecting with it at a critical point is called a secondary path.



### 3.1.3 LINEAR BEHAVIOR

The linear behavior of structures is mathematically represented by linear relations: the fundamental equilibrium path is linear for any value of the load or displacement.

This implies:

- Any load value can be sustained by the structure;
- Critical, turning or failure points are absent in the equilibrium path;
- Structural response can be obtained by exploiting superposition;
- Complete unloading of the structure leads to the reference state.

The above conditions require:

- Perfect linear elasticity behavior for any deformation value;
- Small or infinitesimal deformations;
- Unlimited strength

Despite the difficulties in satisfying the above restrictions (some of them are incompatible each other), the linear model can be usefully used to represent the structure behavior in the vicinity of the reference state.

### 3.1.4 GENERALIZED RESPONSE

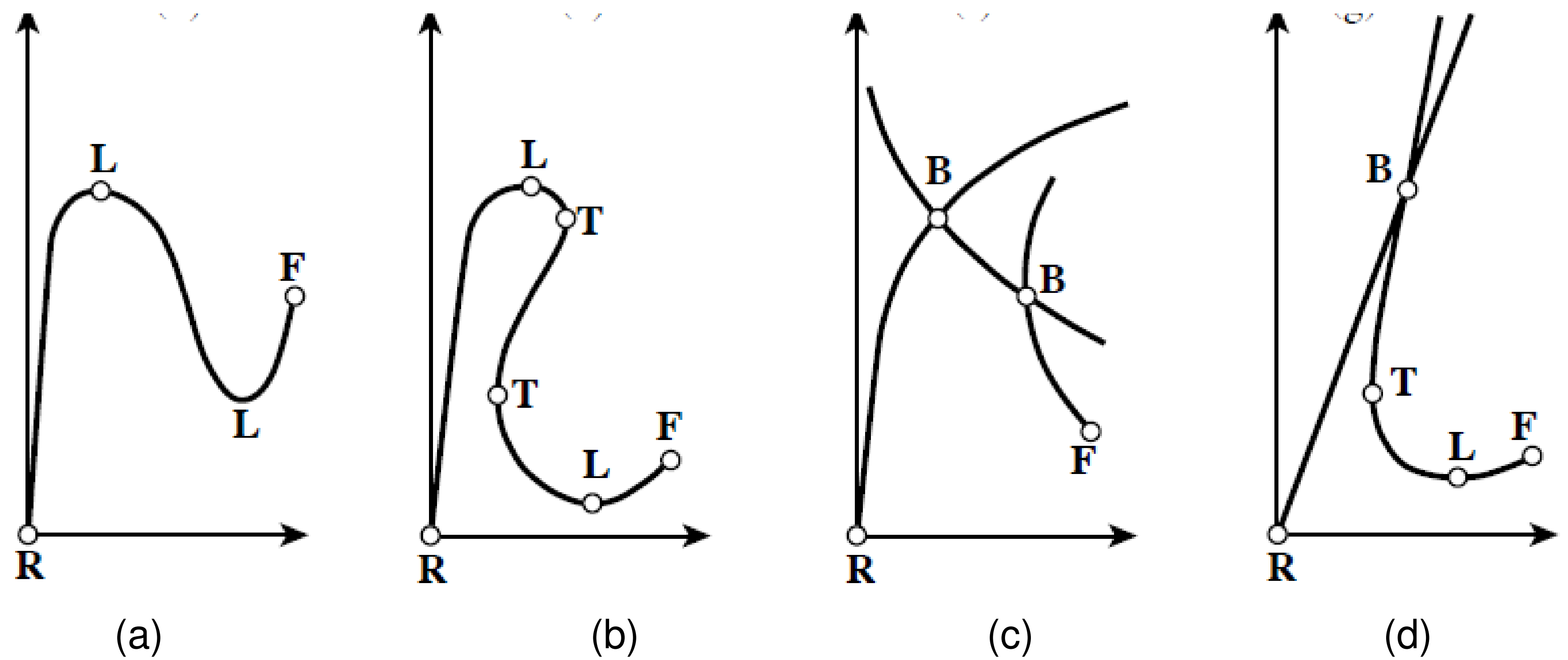
The load-displacement curve requires to introduce:

1. A control parameter, $\lambda$, plotted along the vertical axis, to be represented versus
2. A state parameter, $\mathbf{D}$, plotted along the horizontal axis.

The parameters $\lambda$ and $\mathbf{D}$ characterize in some way the actions applied to the structure and the deformed state of the structure, respectively.

The control parameter is often a load amplitude or load factor or multiplier, while the state parameter is a displacement amplitude of some structure's point.

The well-known load-deflection response is simply a particular case of the control state response.




*Possible response patterns: (a) snap-through, (b) snap-back, (c) bifurcation, (d) bifurcation combined with limit points and snap-back.*

Nonlinear Structural Analysis deals with the prediction of the response of nonlinear structures by model simulation. It involves a combination of mathematical modeling, discretization methods and numerical techniques.

To this end the finite element methods dominate the discretization landscape.

## 3.2 RESIDUAL FORCES

Discrete equilibrium equations in nonlinear static structural analysis are often represented in the compact residual force form:

$$\mathbf{r}(\mathbf{D}, \mathbf{\Lambda}) = \mathbf{0}$$

where:

$\mathbf{r}$ is the residual vector that contains out-of-balance forces,

$\mathbf{D}$ is the state vector (FE nodal displacements that identify the structural configuration),

$\mathbf{\Lambda}$ is an array of control parameters (e.g. commonly mechanical load levels).

The degrees of freedom collected in $\mathbf{D}$ are usually physical or generalized unknown displacements.

It can be also written as:

$$\mathbf{P}(\mathbf{D}) = \mathbf{F}(\mathbf{D}, \boldsymbol{\Lambda}) \quad \text{or} \quad \boxed{\mathbf{r}(\mathbf{D}, \boldsymbol{\Lambda}) = \mathbf{P}(\mathbf{D}) - \mathbf{F}(\mathbf{D}, \boldsymbol{\Lambda}) = \mathbf{0}}$$

where:

$\mathbf{P}$ indicates the internal forces (dependent on the structure's configuration)

$\mathbf{F}$ is the vector of external loads.

If the vector **r** is made to vary with respect to the components of **D** while **Λ** is constant, it provides the Jacobian matrix **K** obtainable upon differentiation:

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{D}} = \mathbf{K}_T, \quad \text{or} \quad K_{ij} = \frac{\partial r_i}{\partial u_j}$$

It is also called the tangent stiffness matrix in structural problems.

### 3.2.1 STAGING IN NONLINEAR ANALYSES

Multiple control parameters are common in real nonlinear problems. They are similar to multiple load conditions in linear problems.

In linear cases, multiple load conditions can be processed independently because any load combination is readily handled by superposition.

In nonlinear problems, however, control parameters can not be varied independently.

Typically, the analysis requires that the user defines the control parameters to be provided to the computer program during the model pre-processing phase.

A **stage** can be defined as "advancing the solution" from $\mathbf{\Lambda}_A$ to $\mathbf{\Lambda}_B$ when the solution $\mathbf{u}_A$ is known.

Moreover, it can be assumed that the <u>components of **Λ** will vary proportionally</u> to a single control parameter λ (stage control parameter) that varies from 0 to 1:

$$\mathbf{\Lambda} = (1-\lambda)\cdot\mathbf{\Lambda}_A + \lambda\cdot\mathbf{\Lambda}_B$$

The nonlinear residual equation to be solved in the stage from $\mathbf{\Lambda}_A$ to $\mathbf{\Lambda}_B$ can be written:

$$\mathbf{r}(\mathbf{D},\lambda) = \mathbf{0}$$

with $\mathbf{D} = \mathbf{D}_A$ for $\lambda = 0$.

<u>The importance of staging in nonlinear static analysis arises from the inapplicability of the superposition principle typical of linear analysis.</u>

For example, the sequences

$$\mathbf{\Lambda}_A \rightarrow \mathbf{\Lambda}_B \rightarrow \mathbf{\Lambda}_C \qquad \mathbf{\Lambda}_A \rightarrow \mathbf{\Lambda}_C$$

do not usually produce the same final solution (intrinsically path-dependent problems).

### 3.2.2 INCREMENTAL FORM OF RESIDUAL EQUATIONS

The incremental form of the residual equation can be obtained by differentiating the residual vector with respect to the time variable which is introduced in order to relate the load factor with a physical or conventional time.

$$\dot{\mathbf{r}}(\mathbf{D},\lambda) = \frac{\partial \mathbf{r}}{\partial \mathbf{D}}\dot{\mathbf{D}} + \frac{\partial \mathbf{r}}{\partial \lambda}\dot{\lambda} = \mathbf{K}\dot{\mathbf{D}} - \mathbf{q}\dot{\lambda} = \mathbf{0}$$

with $\mathbf{K} = \dfrac{\partial \mathbf{r}}{\partial \mathbf{D}} = \mathbf{K}_T, \quad \mathbf{q} = -\dfrac{\partial \mathbf{r}}{\partial \lambda}$ (incremental load vector)

At regular points of the equilibrium path, the stiffness matrix is not singular and the solution provides:

$\dot{\mathbf{D}} = \mathbf{K}^{-1} \cdot \mathbf{q}\dot{\lambda}$ or $\mathbf{v} = \dfrac{\dot{\mathbf{D}}}{\dot{\lambda}} = \dfrac{\partial \mathbf{D}}{\partial t}\dfrac{\partial t}{\partial \lambda} = \mathbf{K}^{-1} \cdot \mathbf{q}$ with $\mathbf{v} = \dfrac{\partial \mathbf{D}}{\partial \lambda} = \mathbf{D}'$

$\mathbf{v}$ being the incremental velocity vector.

### 3.2.3 PROPORTIONAL LOADING

If in the residual equations:

$$\mathbf{r}(\mathbf{D},\lambda)=\mathbf{P}(\mathbf{D})-\mathbf{F}(\cancel{\mathbf{D}},\lambda)=\mathbf{0} \quad \text{or} \quad \mathbf{P}(\mathbf{D})=\mathbf{F}(\cancel{\mathbf{D}},\lambda)$$

the external force vector does not depend on the state parameters (e.g. displacements $\mathbf{D}$), $\underline{\mathbf{F}=\mathbf{F}(\lambda)}$, the above residual relations are called <u>separable</u>.

Moreover if $\mathbf{F}(\lambda)$ is linear in $\lambda$, $\mathbf{F}(\lambda)=\lambda\cdot\mathbf{q}$,

the vector $\mathbf{q}=-\dfrac{\partial\mathbf{r}}{\partial\lambda}=\bar{\mathbf{q}}$ is constant.

The incremental form of the residual equation becomes:

$$\dot{\mathbf{r}}(\mathbf{D},\lambda)=\frac{\partial\mathbf{r}}{\partial\mathbf{D}}\dot{\mathbf{D}}+\frac{\partial\mathbf{r}}{\partial\lambda}\dot{\lambda}=\mathbf{K}\dot{\mathbf{D}}-\bar{\mathbf{q}}\dot{\lambda}=\mathbf{0} \quad \rightarrow \quad \underline{\mathbf{K}\,\dot{\mathbf{D}}=\dot{\lambda}\cdot\bar{\mathbf{q}}}$$

### 3.2.4 CONSERVATIVE SYSTEMS

In this cases the internal force vector $\mathbf{P}$ and the external load vector $\mathbf{F}$ can be obtained by using the corresponding potentials:

$$\mathbf{P} = \frac{\partial U}{\partial \mathbf{D}}, \qquad \mathbf{F} = \frac{\partial W}{\partial \mathbf{D}}$$

or, for the residual force vector: $\mathbf{r} = \dfrac{\partial \Pi}{\partial \mathbf{D}}$

with $\underline{\Pi = U - W}$ being the total potential energy of the system.

The residual equilibrium equations: $\mathbf{r} = \mathbf{0}$ correspond to the fact that – at the equilibrium – the total potential energy is stationary with respect to any variation of the state vector,

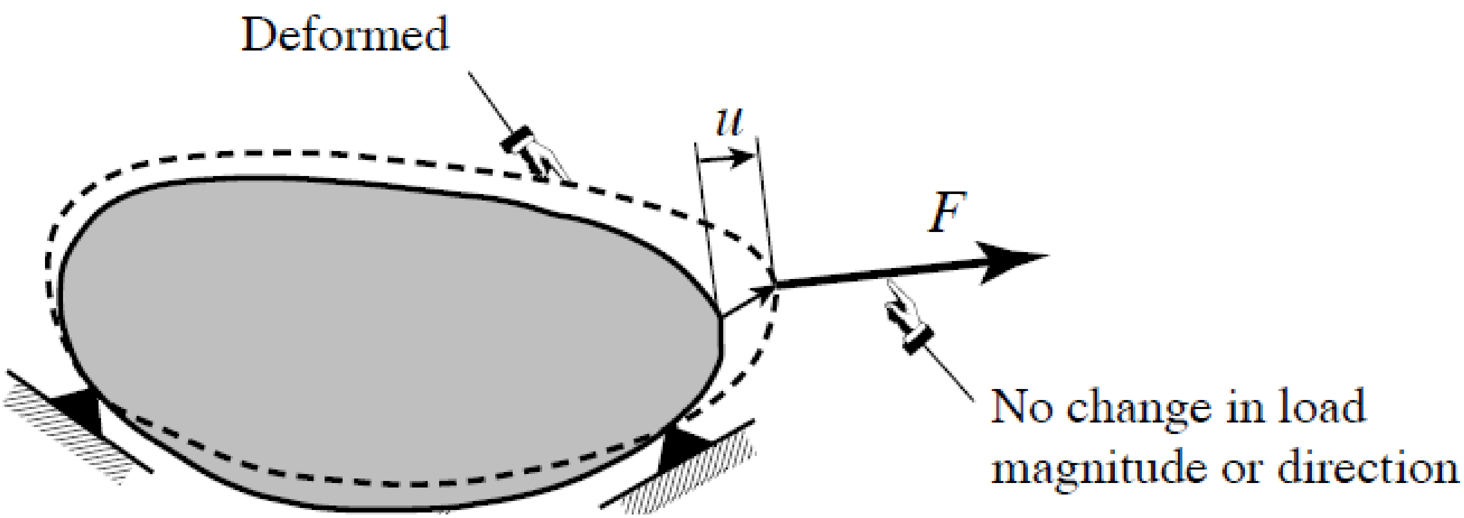
$$\delta\Pi = \mathbf{r}^T \delta\mathbf{D} = \left(\frac{\partial \Pi}{\partial \mathbf{D}}\right)^T \delta\mathbf{D} = 0$$

### a) LOAD POTENTIAL FOR CONSTANT FORCES

The case of constant forces (in magnitude and directions) allows us to simply express the force potential as:

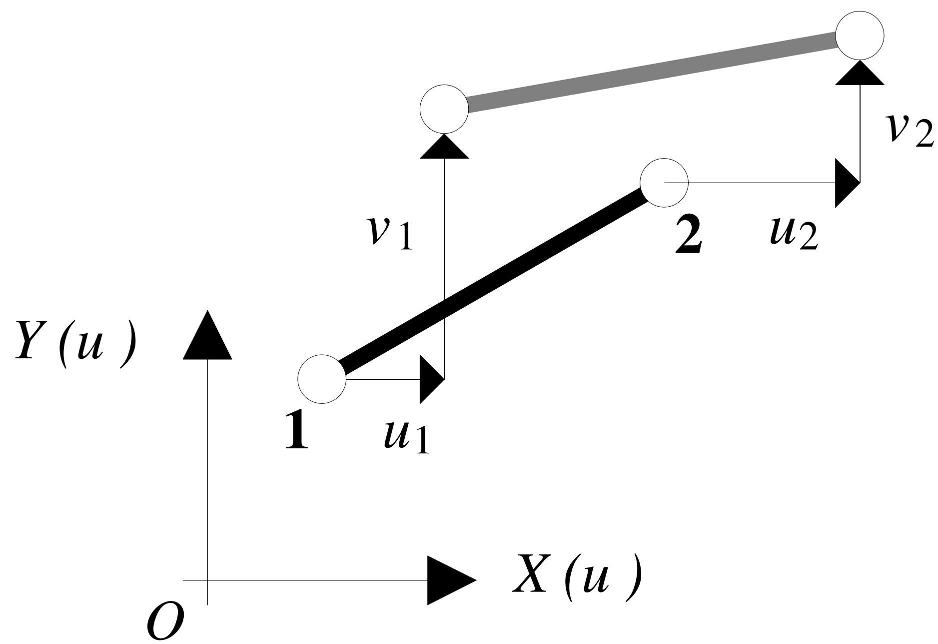
$$W = \sum_{i=1}^{n} \mathbf{F}_i^T \mathbf{D}_i$$

where the vector $\mathbf{D}_i$ indicates the structure deflection at the point of application of the concentrated force $\mathbf{F}_i$.



## b) INTERNAL ENERGY

Consider a simple 2D truss element




The state vector is $\mathbf{D}^T = \{u_1 \quad v_1 \quad u_2 \quad v_2\}$.

The truss elongation is:

$$\delta = l_d - l = \sqrt{(|x_2 - x_1| + u_2 - u_1)^2 + (|y_2 - y_1| + v_2 - v_1)^2} - \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

and the truss internal energy becomes:

$$U = \frac{1}{2} k \delta^2 = \frac{1}{2} k \left[ \begin{array}{l} 2l^2 + 2|x_2 - x_1|(u_2 - u_1) + (u_2 - u_1)^2 + 2|y_2 - y_1|(v_2 - v_1) + (v_2 - v_1)^2 + \\ -2l\sqrt{(|x_2 - x_1| + u_2 - u_1)^2 + (|y_2 - y_1| + v_2 - v_1)^2} \end{array} \right]$$

The components of the internal forces can be obtained as:

$$\mathbf{P} = \frac{\partial U}{\partial \mathbf{D}} = \begin{Bmatrix} \dfrac{\partial U}{\partial u_1} \\ \dfrac{\partial U}{\partial v_1} \\ \dfrac{\partial U}{\partial u_2} \\ \dfrac{\partial U}{\partial v_2} \end{Bmatrix}$$

that are nonlinear functions of the displacements: geometrical nonlinearities have been simply obtained by considering geometric effects that are related to the change of direction of the truss element.

**- Example: 2D bar element (with 2 nodes)**

A simple bar element with linear mechanical behavior is herein assumed (small strains but large displacements and rotations can occur).

By using standard linear interpolation functions (shape functions) the displacement vector of a generic point along the bar can be written as:

$$\mathbf{D}(\xi)=\begin{Bmatrix} u(\xi) \\ v(\xi) \end{Bmatrix}=\begin{bmatrix} \frac{1}{2}(1-\xi) & 0 & \frac{1}{2}(1+\xi) & 0 \\ 0 & \frac{1}{2}(1-\xi) & 0 & \frac{1}{2}(1+\xi) \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}=\mathbf{N}(\xi)\,\mathbf{D}$$

where the natural coordinate $-1 \le \xi \le 1$ span the truss element (parent element).

In the Total Lagrangian formulation the Green-Lagrange (GL) strains $\mathsf{E}$ and the second Piola-Kirchhoff stresses $S$ are used.

The only GL strain component in this case is: $\mathsf{E} = \frac{1}{2}\frac{(L^2 - L_0^2)}{L_0^2} = const$

It can be written as:

$$\mathsf{E} = \mathsf{E}_l + \mathsf{E}_n = \left[\mathbf{B}_l + \mathbf{B}_n(\mathbf{u})\right]\mathbf{D} = \mathbf{B}\ \mathbf{D}$$

where the linear ($\mathbf{B}_l$) and nonlinear ($\mathbf{B}_n$) compatibility matrices have been used.

Initial and final length of the bar:

$$L_0^2 = (X2 - X1)^2 + (Y2 - Y1)^2$$

$$L^2 = (x2 - x1)^2 + (y2 - y1)^2$$

Current coordinates of the nodes1 and 2:

$$x1 := X1 + u1$$

$$y1 := Y1 + v1$$

$$x2 := X2 + u2$$

$$y2 := Y2 + v2$$

## Green Lagrange strain:

$$E = \frac{\left[(X2+u2-(X1+u1))^2+(Y2+v2-(Y1+v1))^2\right]-\left[(X2-X1)^2+(Y2-Y1)^2\right]}{2\cdot L0^2}$$

$$\frac{1}{2}\cdot\frac{\left[(X2+u2-X1-u1)^2+(Y2+v2-Y1-v1)^2-(X2-X1)^2-(Y2-Y1)^2\right]}{L0^2}$$

$$\frac{-1}{2}\cdot\frac{\left(-2\cdot X2\cdot u2+2\cdot X2\cdot u1-u1^2-2\cdot Y2\cdot v2+2\cdot Y2\cdot v1-v2^2-u2^2+2\cdot u2\cdot X1+2\cdot u2\cdot u1+2\cdot v2\cdot Y1+2\cdot v2\cdot v1-2\cdot Y1\cdot v1-v1^2-2\cdot X1\cdot u1\right)}{L0^2}$$

$$\frac{-X2+\frac{1}{2}\cdot u1-\frac{u2}{2}+X1}{L0^2}\cdot u1 \qquad \frac{-Y2+\frac{1}{2}\cdot v1-\frac{v2}{2}+Y1}{L0^2}\cdot v1 \qquad \frac{X2+\frac{1}{2}\cdot u2-X1-\frac{u1}{2}}{L0^2}\cdot u2 \qquad \frac{Y2+\frac{1}{2}\cdot v2-Y1-\frac{v1}{2}}{L0^2}\cdot v2$$

$$E_l = \frac{\left(-(X2-X1)\quad -(Y2-Y1)\quad (X2-X1)\quad (Y2-Y1)\right)}{L0^2}\cdot\begin{bmatrix}u1\\v1\\u2\\v2\end{bmatrix} = BL\cdot\begin{pmatrix}u1\\v1\\u2\\v2\end{pmatrix}$$

Depends on the bar's initial configuration only

$$E_n = \frac{\left(\frac{u1-u2}{2}\quad \frac{v1-v2}{2}\quad \frac{u2-u1}{2}\quad \frac{v2-v1}{2}\right)}{L0^2}\cdot\begin{bmatrix}u1\\v1\\u2\\v2\end{bmatrix} = \frac{1}{2}\cdot(u1\quad v1\quad u2\quad v2)\cdot BN\cdot\begin{pmatrix}u1\\v1\\u2\\v2\end{pmatrix}$$

Depends on displacements

Stress state (second Piola-Kirchhoff stress) in the axial direction:

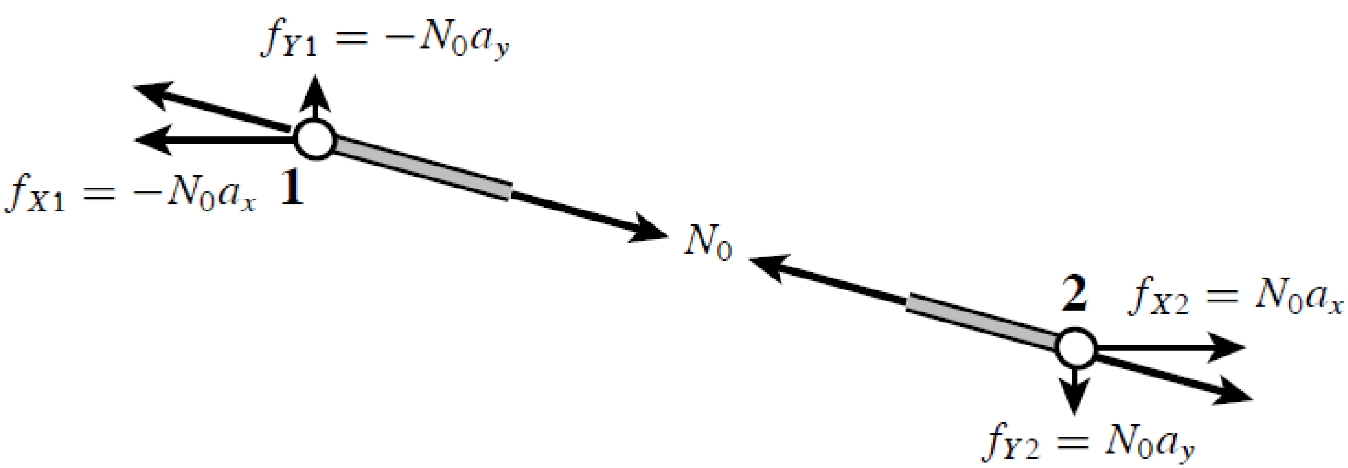
$$S = S_0 + E \cdot \mathsf{E} \qquad \text{and related axial force:} \qquad N = A_0\, S$$

$S_0$ : initial stress present in the bar (if any)

Case of external conservative and proportional loads: $\mathbf{F} = \lambda \mathbf{q}$

Total potential energy:

$$\Pi = U - W = \int_{V_0} (S_0 \mathsf{E} + \frac{1}{2} E \mathsf{E}^2) dV_o - \mathbf{F}^T \mathbf{D} = \int_{L_0} A_0 (S_0 \mathsf{E} + \frac{1}{2} E \mathsf{E}^2) dX - \lambda \mathbf{q}^T \mathbf{D}$$




$$\mathbf{p} = N L_0 \mathbf{B}^T$$

The stationary condition becomes ( $\delta \mathsf{E} = \mathbf{B}\,\delta\mathbf{D} = (\mathbf{B}_l + \mathbf{B}_n)\,\delta\mathbf{D}$ ):

$$\delta\Pi = \delta U - \delta W = \int_{L_0} A_0 (S_0 \delta\mathsf{E} + E\mathsf{E}\,\delta\mathsf{E})\,dX - \lambda \mathbf{q}^T \delta\mathbf{D} = N_0 L_0 \mathbf{B}\,\delta\mathbf{D} - \lambda\mathbf{q}^T \delta\mathbf{D} = 0$$

$$\delta U = \int_{L_0} A_0 (S_0 \underbrace{\delta\mathsf{E}}_{\mathbf{B}\,\delta\mathbf{D}} + E\mathsf{E}\underbrace{\delta\mathsf{E}}_{\mathbf{B}\,\delta\mathbf{D}})\,dX = \underbrace{N\;L_0\mathbf{B}}_{\mathbf{P}^T}\delta\mathbf{D} = \mathbf{P}^T \delta\mathbf{D}$$ with $S = S_0 + E\cdot\mathsf{E}$

$\mathbf{P} = N\,L_0 \mathbf{B}^T$ : internal force vector, with $N = A_0\; S = A_0 (S_0 + E\mathsf{E})$

The tangent stiffness matrix can be written as:

$$\mathbf{K}_T = \frac{\partial\mathbf{P}}{\partial\mathbf{D}} = \frac{\partial (N L_0 \mathbf{B}^T)}{\partial\mathbf{D}} = \underbrace{\left( A_0 L_0 \mathbf{B}^T \frac{\partial S}{\partial\mathbf{D}} \right)}_{\mathbf{K}_M} + \underbrace{\left( A_0 L_0 S \frac{\partial\mathbf{B}^T}{\partial\mathbf{D}} \right)}_{\mathbf{K}_G} = \mathbf{K}_M + \mathbf{K}_G$$

$\mathbf{K}_M$ material tangent stiffness matrix

$\mathbf{K}_G$ geometric tangent stiffness matrix

Being: $\dfrac{\partial S}{\partial \mathbf{D}} = \dfrac{\partial (S_0 + E\mathsf{E})}{\partial \mathbf{D}} = E\dfrac{\partial \mathsf{E}}{\partial \mathbf{D}} = E\mathbf{B}$ with $\mathsf{E} = \mathbf{B}\,\mathbf{D}$, $\mathbf{B} = \mathbf{B}_l + \mathbf{B}_n(\mathbf{D})$

where

$$\mathbf{K}_M = A_0 L_0 \mathbf{B}^T \frac{\partial S}{\partial \mathbf{D}} = A_0 L_0 \mathbf{B}^T \frac{\partial \overbrace{(S_0 + E\mathbf{BD})}^{S}}{\partial \mathbf{D}} = E A_0 L_0 \mathbf{B}^T \mathbf{B}$$

The material tangent stiffness matrix $\mathbf{K}_M$ depends only on the material properties (for $E$ use the tangent elastic modulus of the material, $E_T$ ).

Finally:

$$\mathbf{K}_G = A_0 L_0 S \frac{\partial \mathbf{B}^T}{\partial \mathbf{D}} = \frac{A_0 S}{L_0}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \frac{N}{L_0}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The geometric tangent stiffness matrix $\mathbf{K}_G$ depends only on the stress state $S$ .

# 3.3 SOLUTION OF NON-LINEAR PROBLEMS

The solution of the residual equations:

$$\mathbf{r} = \mathbf{P}(\mathbf{D}) - \mathbf{F} = \mathbf{0} \quad \text{or} \quad \mathbf{P}(\mathbf{D}) = \mathbf{F}$$

can be obtained by bringing to zero the residual force vector $\mathbf{r}$ (in the following indicated with $\psi$ for the discretized structure), while the displacement vector is $\mathbf{D}$):

$$\psi = \mathbf{P}(\mathbf{D}) - \mathbf{F} \neq \mathbf{0}$$

The requirement $\psi \rightarrow \mathbf{0}$ must be fulfilled by mean of an iterative process.

The vector $\psi \neq \mathbf{0}$ can be interpreted as a measure of the equilibrium violation.

The generic term of the unbalanced force vector can be written through a first order power series expansion as:

$$\psi_i^k = -\sum_{j=1}^{N} \left( \frac{\partial \psi_i}{\partial \mathbf{D}_j} \right)^k d\mathbf{D}_j^k$$

In matrix form we have:

$$\boldsymbol{\psi}(\mathbf{D}^k) = -\mathbf{J}(\mathbf{D}^k) \cdot d\mathbf{D}^k$$

where the Jacobian matrix $J_{ij}$ is defined as:

$$J_{ij} = \left( \frac{\partial \psi_i}{\partial D_j} \right)^k = \frac{\partial \boldsymbol{\psi}(\mathbf{D}^k)}{\partial \mathbf{D}} = K_{ij}^k + \cancel{\sum_{l=1}^{N} \left( \frac{\partial K_{il}}{\partial D_j} \right) \cdot D_n^k}$$

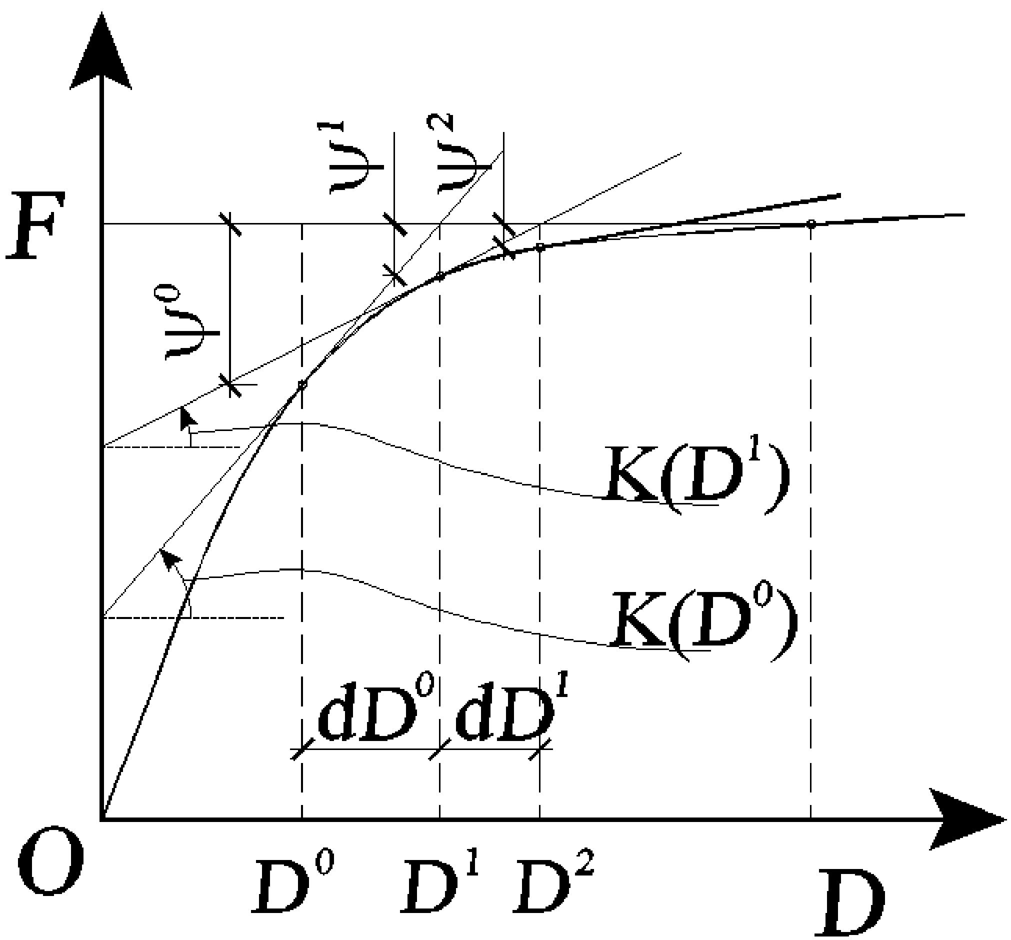
The second term corresponds to a higher order contributions and can be neglected

The algebraic system becomes:

$$\mathbf{K}(\mathbf{D}^k) \cdot d\mathbf{D}^k = -\boldsymbol{\psi}(\mathbf{D}^k) = -\boldsymbol{\psi}^k$$

The matrix $\mathbf{K}(\mathbf{D}^k)$ is the tangent stiffness matrix, i.e. its the local gradient of the force-displacements relationship.

The solution strategy based on the **tangent stiffness method** (**TSM or Newton's method**) is illustrated in the following figure for a 1D problem.



The method operates by:

- <u>first attempt of the solution</u>: $\mathbf{D}^0$ (often it is assumed $\mathbf{D}^0 = \mathbf{0}$);
- determine the corresponding <u>initial tangent stiffness matrix</u>: $\mathbf{K}(\mathbf{D}^0)$;
- determine the <u>initial vector of unbalanced forces</u>: $\boldsymbol{\psi}(\mathbf{D}^0) = \mathbf{K}(\mathbf{D}^0) \cdot \mathbf{D}^0 - \mathbf{F} \neq 0$
- solve for $d\mathbf{D}^0$:

$$d\mathbf{D}^0 = -\mathbf{K}^{-1}(\mathbf{D}^0) \cdot \boldsymbol{\psi}(\mathbf{D}^0)$$

(for some choice of $\mathbf{D}^0$ the matrix $\mathbf{K}(\mathbf{D}^0)$ may not be invertible!).

- update the displacement vector: $\mathbf{D}^1 = \mathbf{D}^0 + d\mathbf{D}^0$
- determine $\mathbf{K}(\mathbf{D}^1), \boldsymbol{\psi}(\mathbf{D}^1)$ and solve for $d\mathbf{D}^0$:

$$d\mathbf{D}^1 = -\mathbf{K}^{-1}(\mathbf{D}^1) \cdot \boldsymbol{\psi}(\mathbf{D}^1)$$

- Continue up to the fulfillment of some <u>convergence criteria</u>.

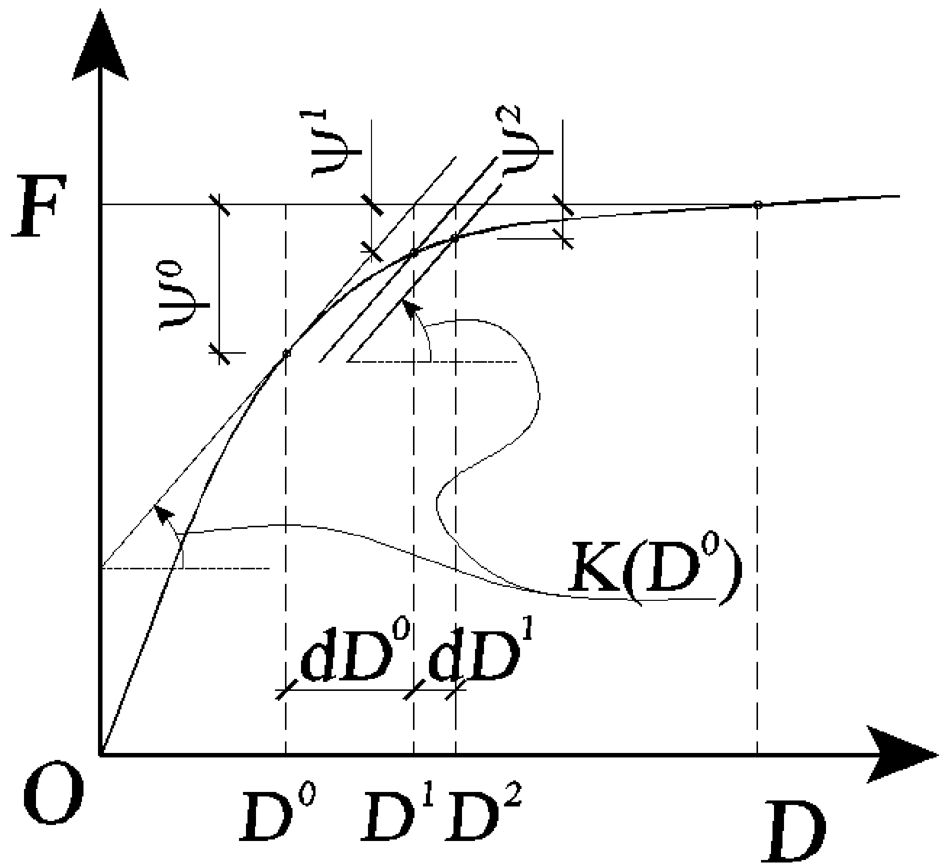
The stiffness matrix must be inverted at each iteration (by using the Gauss algorithm the inversion of a square nxn matrix requires a number of operations of the order of $O(n^3/3)$).

The **initial stiffness method** (**ISM or modified Newton-Rapson**) does not require to invert the matrix at every iteration.

The initial stiffness matrix is used throughout the computational process.

The recursive solution operation becomes:

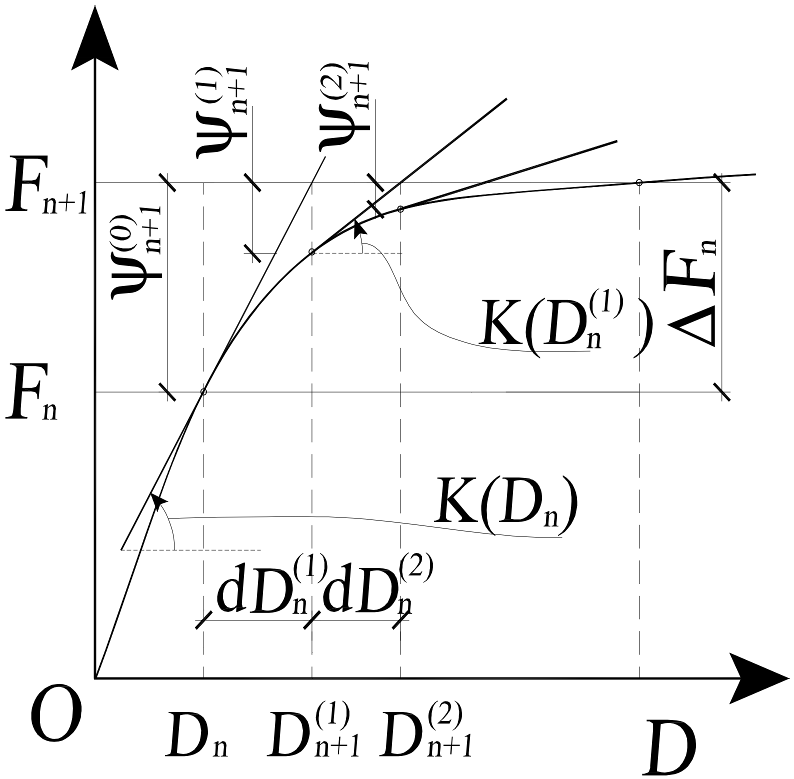
$$d\mathbf{D}^k = -\mathbf{K}^{-1}(\mathbf{D}^0)\cdot\boldsymbol{\psi}(\mathbf{D}^k)$$

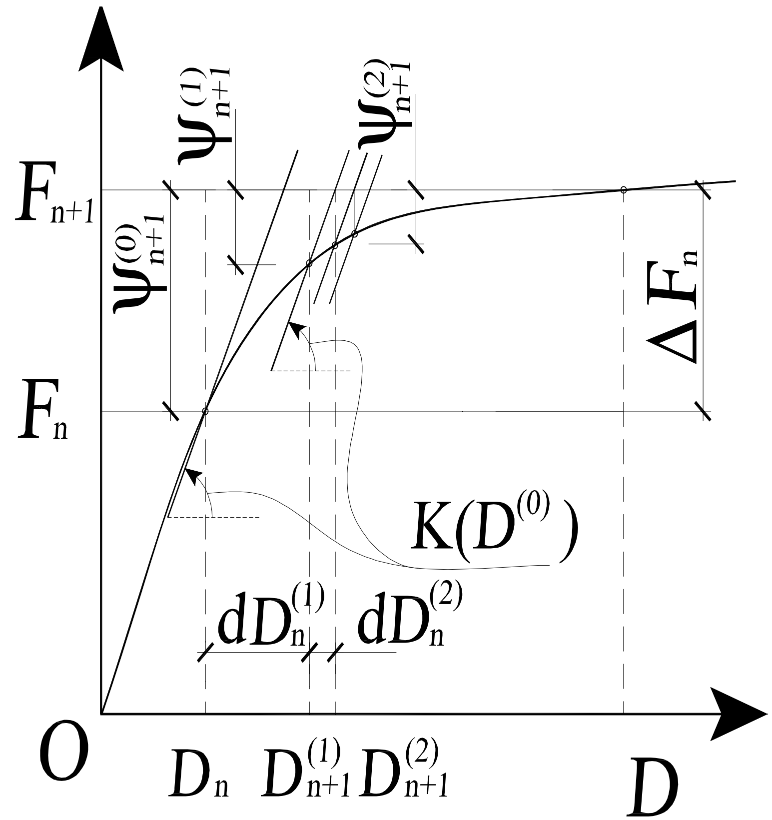



This method usually requires a greater number of iterations than the TSM.

This approach is unconditionally stable and can be used also for softening materials.

These methods can be refined by applying the loads to the structure in more than one single step; this aspect is particularly important when the structural response depends of the load path, such as for plasticity problems.




The method of the **tangent stiffness** applied in <u>one single iteration</u> for each load step, corresponds to the <u>direct integration</u> of the problem.

By applying the load in several steps we have:

$$\mathbf{P}(\mathbf{D}) = \lambda \cdot \mathbf{F}_0$$

where $\lambda$ is the load factor and $\mathbf{F}_0$ the base vector of the nodal forces.

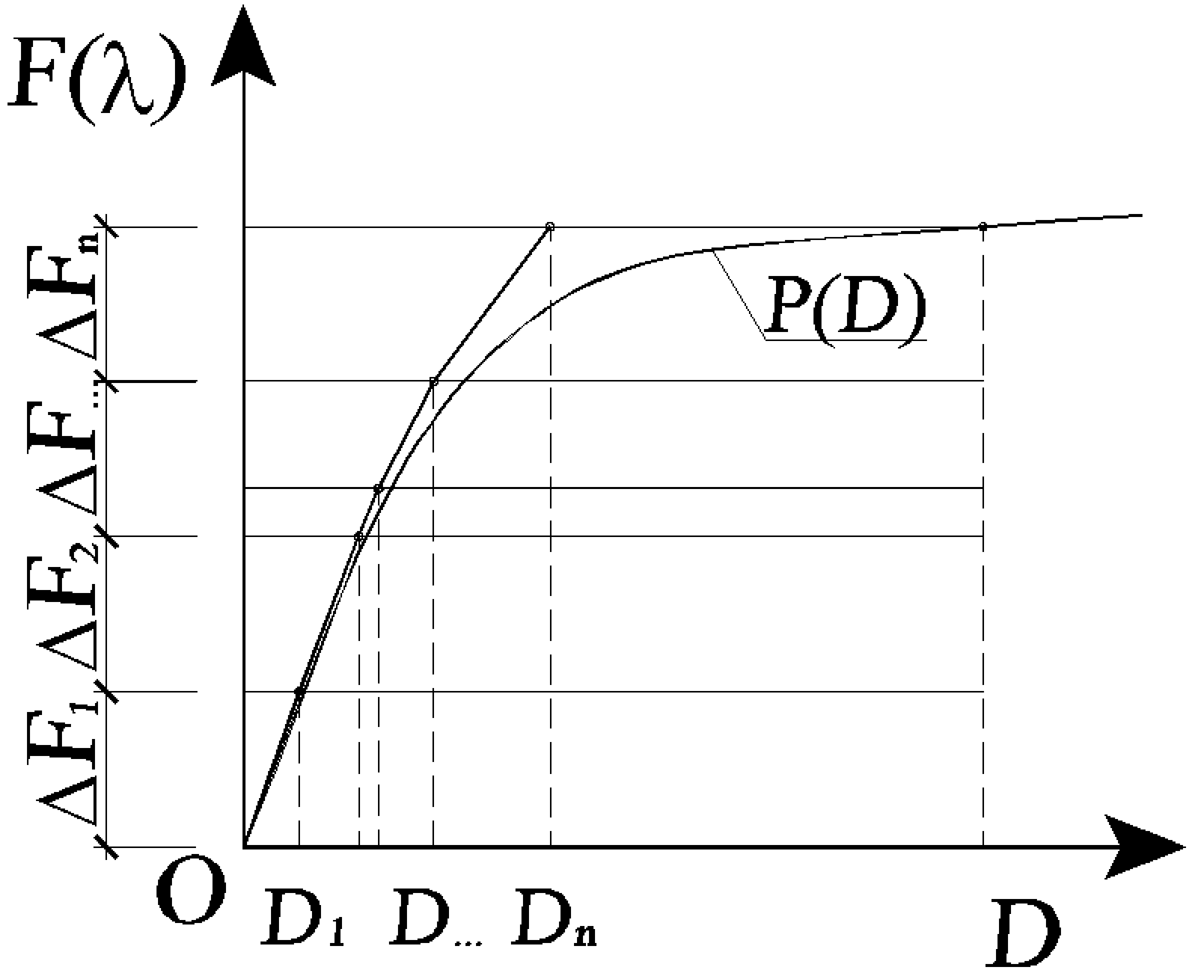
By differentiating with respect to $\lambda$:

$$\frac{d\mathbf{P}}{d\mathbf{D}} \frac{d\mathbf{D}}{d\lambda} = \mathbf{K}_T \cdot \frac{d\mathbf{D}}{d\lambda} = \mathbf{F}_0$$

$$\frac{d\mathbf{D}}{d\lambda} = \mathbf{K}_T^{-1} \cdot \mathbf{F}_0 , \qquad d\mathbf{D}^j = d\lambda \cdot \mathbf{K}_T^{-1} \cdot \mathbf{F}_0$$

The incremental displacements are evaluated with respect to the incremental load factor multiplier $d\lambda$, with $\mathbf{K}_T^{-1} = \left(d\mathbf{P} / d\mathbf{D}\right)^{-1}$

Such an approach is usually <u>divergent</u> if the load is not applied in a sufficiently high number of steps (small values for $d\lambda$).




**NB: All the above methods cannot follow the equilibrium path when <u>snap-back</u> or <u>snap-through</u> occur.**

# 3.4 DETAILS OF THE SOLUTION PROCEDURES

## 3.4.1 NEWTON-RAPSON METHOD

- Most popular method
- Assume $\mathbf{d}^{i}$ at i-th iteration is known
- Looking for $\mathbf{d}^{i+1}$ from first-order Taylor series expansion

$$\mathbf{P}(\mathbf{d}^{i+1}) \approx \mathbf{P}(\mathbf{d}^{i}) + \mathbf{K}_{T}^{i}(\mathbf{d}^{i}) \cdot \Delta\mathbf{d}^{i} = \mathbf{F}$$

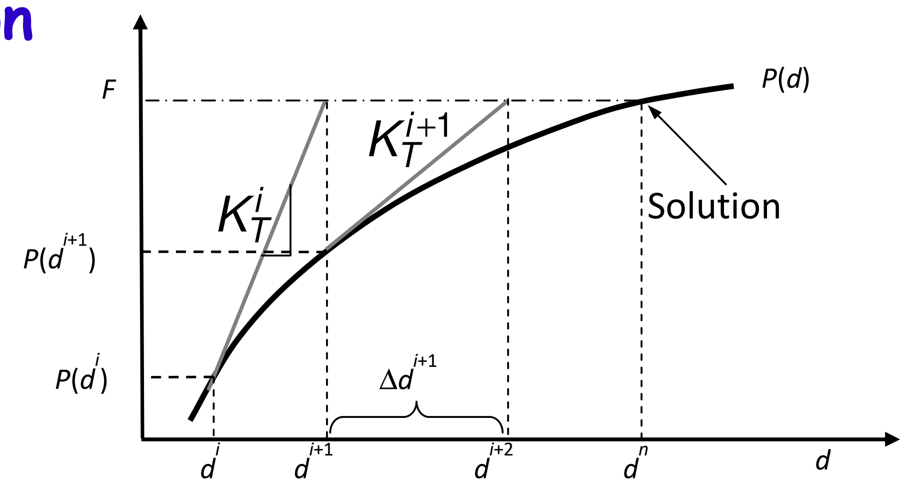
$$\mathbf{K}_{T}^{i}(\mathbf{d}^{i}) \equiv \left(\frac{\partial \mathbf{P}}{\partial \mathbf{d}}\right)^{i}$$

: **Jacobian matrix or Tangent stiffness matrix**

- **Solve for incremental solution**
- Update solution

$$\mathbf{K}_{T}^{i}\Delta\mathbf{d}^{i} = \mathbf{F} - \mathbf{P}(\mathbf{d}^{i})$$

$$\mathbf{d}^{i+1} = \mathbf{d}^{i} + \Delta\mathbf{d}^{i}$$



**N-R METHOD CONT.**

- Observations:
  - **Second-order convergence** near the solution (Fastest method!)

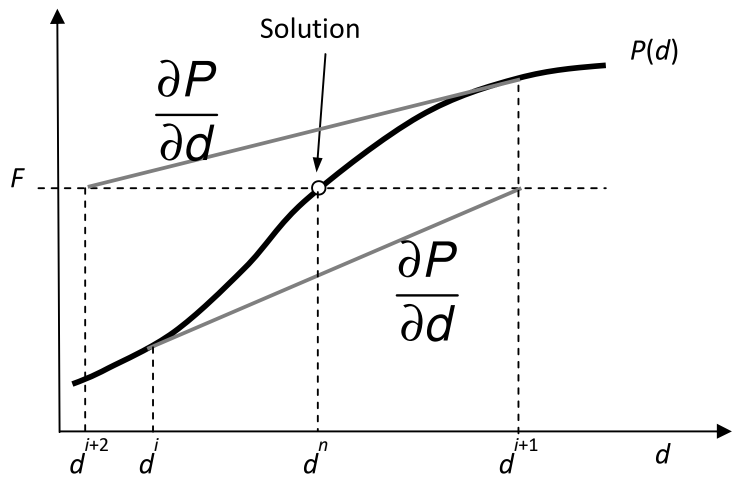
$$\lim_{n\to\infty}\frac{\left|u_{exact}-u_{n+1}\right|}{\left|u_{exact}-u_{n}\right|^2}=c$$

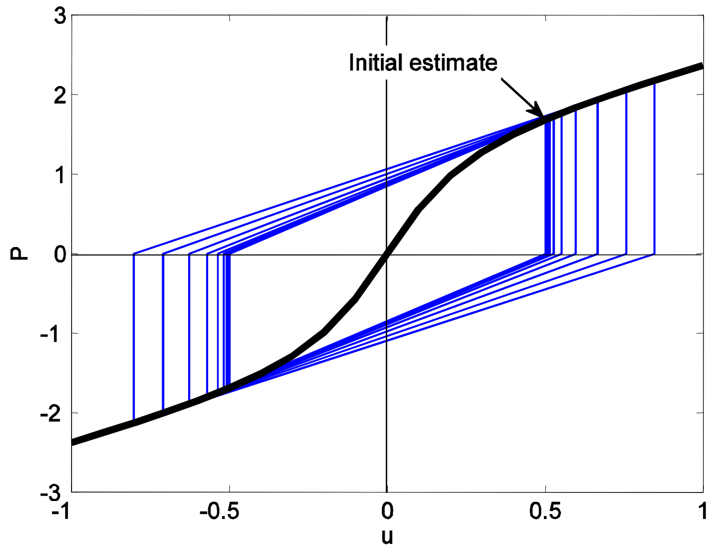
  - Tangent stiffness $\mathbf{K}_T^i(\mathbf{d}^i)$ is not constant
  - The matrix equation solves for incremental displacement $\Delta\mathbf{d}^i$
  - RHS is not a force but a **residual force,** $\mathbf{R}^i \equiv \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$
  - Iteration stops when conv < tolerance

$$\text{conv} = \frac{\sum_{j=1}^{n}(R_j^{i+1})^2}{1+\sum_{j=1}^{n}(F_j)^2} \qquad \text{Or,} \qquad \text{conv} = \frac{\sum_{j=1}^{n}(\Delta u_j^{i+1})^2}{1+\sum_{j=1}^{n}(\Delta u_j^{0})^2}$$

**IN SOME CASES THE N-R METHOD DOES NOT CONVERGE**

- Difficulties
  - **Convergence is not always guaranteed**
  - Automatic load step control and/or line search techniques are often used
  - Difficult/expensive to calculate $\mathbf{K}_T^i(\mathbf{d}^i)$



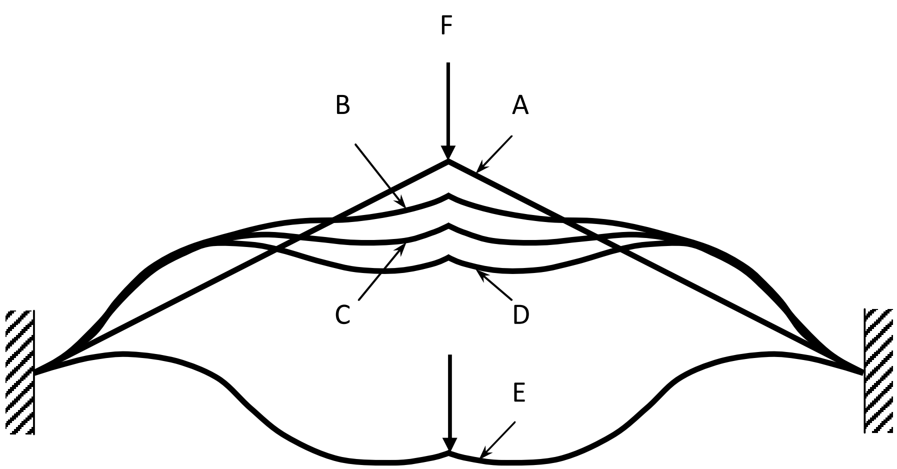



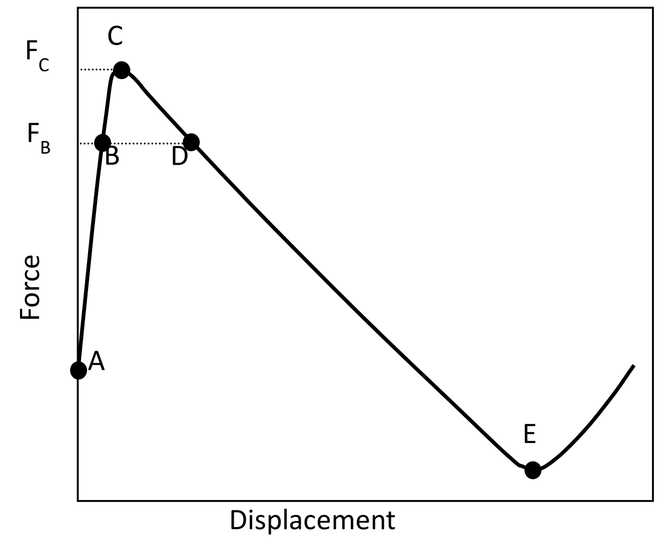
**WHEN N-R METHOD DOES NOT CONVERGE CONT.**

- Convergence difficulty occurs when
Jacobian matrix is not positive-definite

P.D. Jacobian: in order to increase displ., force must be increased

Bifurcation & snap-through require a special algorithm



### 3.4.2 MODIFIED N-R METHOD

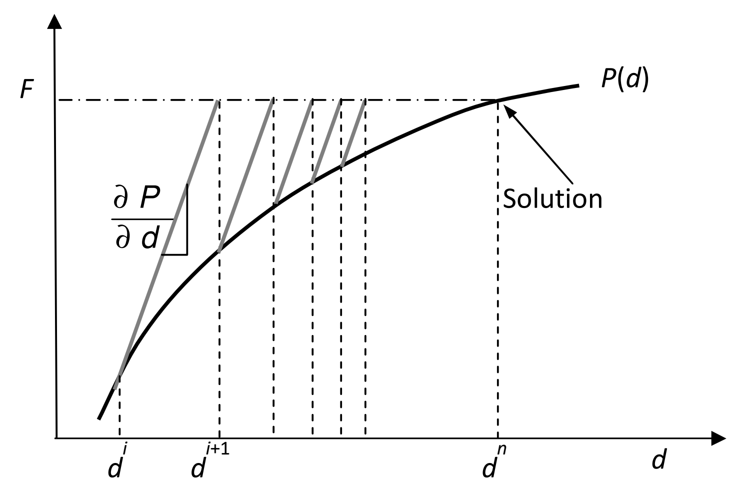
- Constructing $\mathbf{K}_T^i(\mathbf{d}^i)$ and solving $\mathbf{K}_T^i \Delta \mathbf{d}^i = \mathbf{R}^i$ is expensive
- Computational Costs (Let the matrix size be N x N)

L-U factorization ~ $N^3$

Forward/backward substitution ~ N

- **Use L-U factorized $\mathbf{K}_T^i(\mathbf{d}^i)$ repeatedly**
- More iteration is required, but

  each iteration is fast
- **More stable than N-R method**
- Hybrid N-R method can be used



**EXAMPLE – MODIFIED N-R METHOD**

- Solve the same problem using modified N-R method

$$\mathbf{P}(\mathbf{d}) \equiv \begin{Bmatrix} d_1 + d_2 \\ d_1^2 + d_2^2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 9 \end{Bmatrix} \equiv \mathbf{F} \qquad \mathbf{d}^0 = \begin{Bmatrix} 1 \\ 5 \end{Bmatrix} \qquad \mathbf{P}(\mathbf{d}^0) = \begin{Bmatrix} 6 \\ 26 \end{Bmatrix}$$

$$\mathbf{K}_T = \frac{\partial \mathbf{P}}{\partial \mathbf{d}} = \begin{bmatrix} 1 & 1 \\ 2d_1 & 2d_2 \end{bmatrix} \qquad \mathbf{R}^0 = \mathbf{F} - \mathbf{P}(\mathbf{d}^0) = \begin{Bmatrix} -3 \\ -17 \end{Bmatrix}$$

- Iteration 1

$$\begin{bmatrix} 1 & 1 \\ 2 & 10 \end{bmatrix} \begin{Bmatrix} \Delta d_1^0 \\ \Delta d_2^0 \end{Bmatrix} = \begin{Bmatrix} -3 \\ -17 \end{Bmatrix} \Longrightarrow \begin{Bmatrix} \Delta d_1^0 \\ \Delta d_2^0 \end{Bmatrix} = \begin{Bmatrix} -1.625 \\ -1.375 \end{Bmatrix}$$

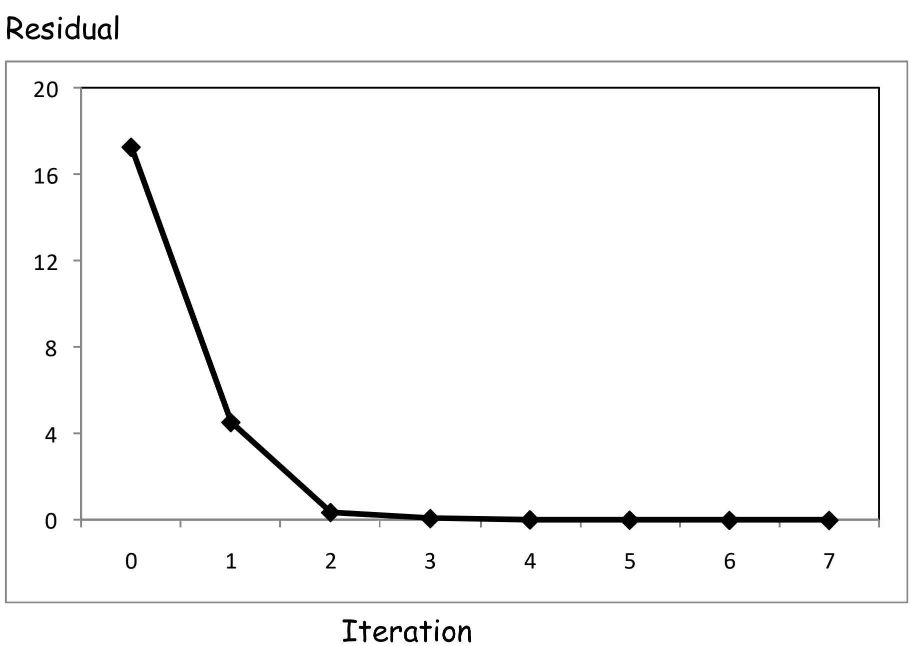
$$\mathbf{d}^1 = \mathbf{d}^0 + \Delta \mathbf{d}^0 = \begin{Bmatrix} -0.625 \\ 3.625 \end{Bmatrix} \qquad \mathbf{R}^1 = \mathbf{F} - \mathbf{P}(\mathbf{d}^1) = \begin{Bmatrix} 0 \\ -4.531 \end{Bmatrix}$$

## EXAMPLE – MODIFIED N-R METHOD CONT.

- Iteration 2

$$\begin{bmatrix} 1 & 1 \\ 2 & 10 \end{bmatrix} \begin{Bmatrix} \Delta d_1^1 \\ \Delta d_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4.531 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \Delta d_1^1 \\ \Delta d_2^1 \end{Bmatrix} = \begin{Bmatrix} 0.566 \\ -0.566 \end{Bmatrix}$$

$$\mathbf{d}^2 = \mathbf{d}^1 + \Delta\mathbf{d}^1 = \begin{Bmatrix} -0.059 \\ 3.059 \end{Bmatrix} \qquad \mathbf{R}^2 = \mathbf{F} - \mathbf{P}(\mathbf{d}^2) = \begin{Bmatrix} 0 \\ -0.358 \end{Bmatrix}$$

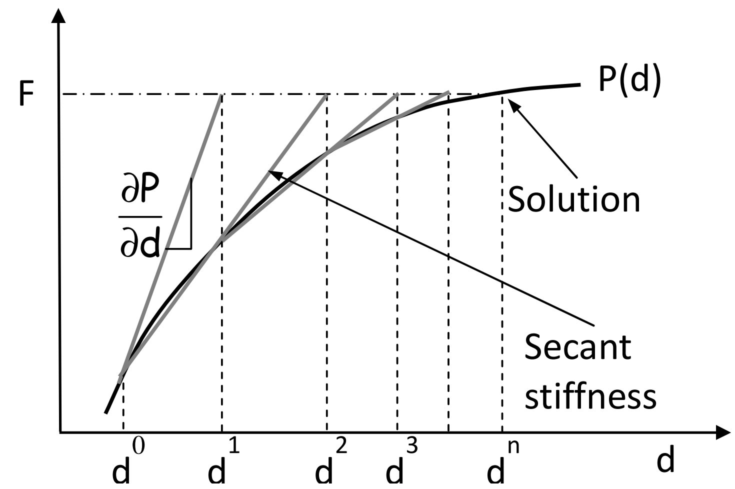



| Iter | \|\|R\|\| |
|---|---|
| 0 | 17.263 |
| 1 | 4.5310 |
| 2 | 0.3584 |
| 3 | 0.0831 |
| 4 | 0.0204 |
| 5 | 0.0051 |
| 6 | 0.0013 |
| 7 | 0.0003 |

### 3.4.3 INCREMENTAL SECANT METHOD

- <u>Secant matrix</u>

- Instead of using tangent stiffness, **approximate it** using the solution from the previous iteration
- At i-th iteration




$$\mathbf{K}_s^i \Delta \mathbf{d}^i = \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$$

- The secant matrix satisfies

$$\mathbf{K}_s^i \cdot (\mathbf{d}^i - \mathbf{d}^{i-1}) = \mathbf{P}(\mathbf{d}^i) - \mathbf{P}(\mathbf{d}^{i-1})$$

- Not a unique process in high dimension

- **Start from initial $K_T$ matrix, iteratively update it**

- Rank-1 or rank-2 update
- The textbook has Broyden's algorithm (Rank-1 update)
- Here we will discuss BFGS method (Rank-2 update)

- BFGS (Broyden, Fletcher, Goldfarb and Shanno) method

Stiffness matrix must be symmetric and positive-definite

$$\Delta \mathbf{d}^i = [\mathbf{K}_s^i]^{-1}\{\mathbf{F} - \mathbf{P}(\mathbf{d}^i)\} \equiv [\mathbf{H}_s^i]\{\mathbf{F} - \mathbf{P}(\mathbf{d}^i)\}$$

Instead of updating **K**, update **H** (saving computational time)

$$\mathbf{H}_s^i = (\mathbf{I} + \mathbf{w}^i \mathbf{v}^{iT})\mathbf{H}_s^{i-1}(\mathbf{I} + \mathbf{w}^i \mathbf{v}^{iT})$$

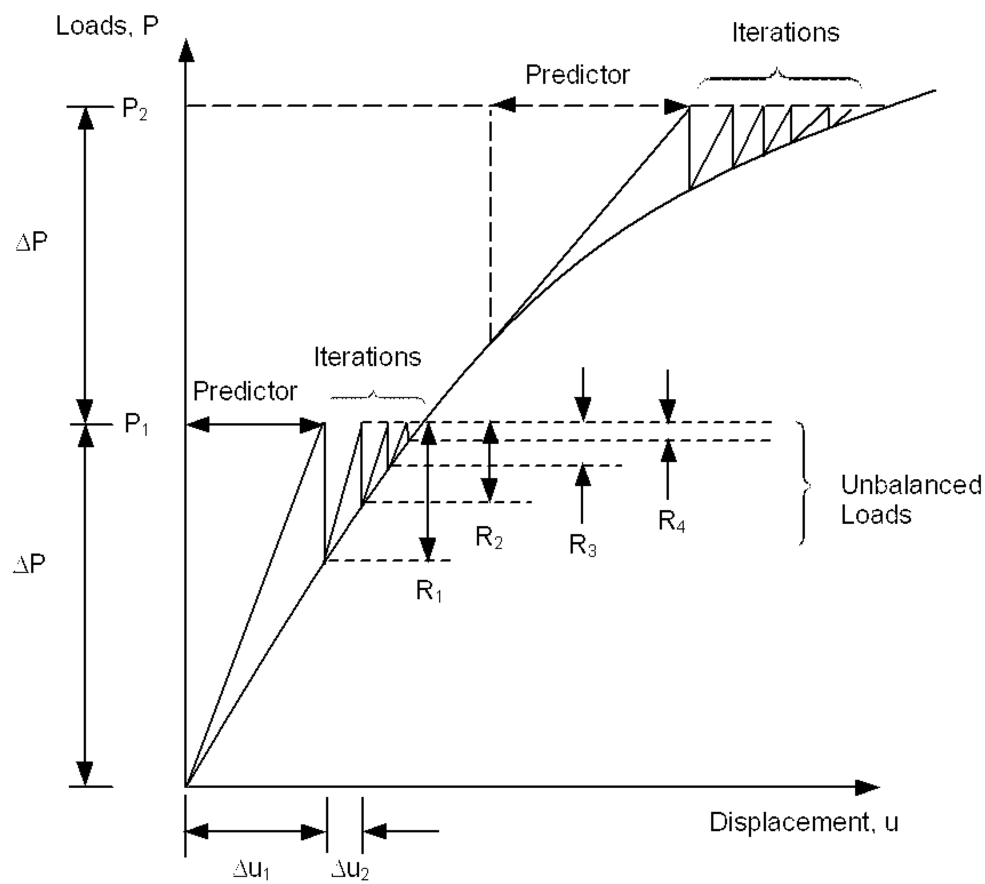
$$\mathbf{v}^i = \mathbf{R}^{i-1}\left(1 - \frac{(\Delta \mathbf{d}^{i-1})^T(\mathbf{R}^{i-1} - \mathbf{R}^i)}{(\Delta \mathbf{d}^i)^T \mathbf{R}^{i-1}}\right) - \mathbf{R}^i$$

$$\mathbf{w}^i = \frac{\Delta \mathbf{d}^{i-1}}{(\Delta \mathbf{d}^{i-1})^T(\mathbf{R}^{i-1} - \mathbf{R}^i)}$$

Become unstable when the No. of iterations is increased

### 3.4.4 INCREMENTAL FORCE METHOD

- N-R method converges fast if the initial estimate is close to the solution
- Solid mechanics: initial estimate = undeformed shape
- Convergence difficulty occurs when the applied load is large (deformation is large)
- <u>IFM: apply loads in increments</u>. Use the solution from the previous increment as an initial estimate
- Commercial programs call it "**Load Increment**" or "**Time Increment**"



## INCREMENTAL FORCE METHOD CONT.

- Load increment does not have to be uniform
  - Critical part has smaller increment size

- Solutions in the intermediate load increments
  - History of the response can provide insight into the problem
  - Estimating the bifurcation point or the critical load
  - Load increments greatly affect the accuracy in path-dependent problems

### 3.4.5 LOAD INCREMENT IN COMMERCIAL SOFTWARE

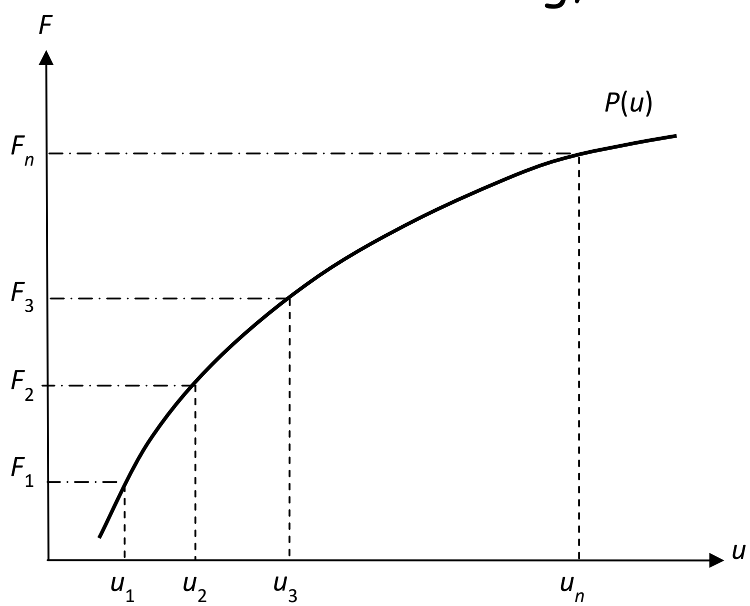
- Use "Time" to represent load level
  - In a static problem, "Time" means a pseudo-time
  - Required Starting time, ($T_{start}$), Ending time ($T_{end}$) and time increment
  - Load is gradually increased from zero at $T_{start}$ and full load at $T_{end}$
  - Load magnitude at load increment $T^n$:

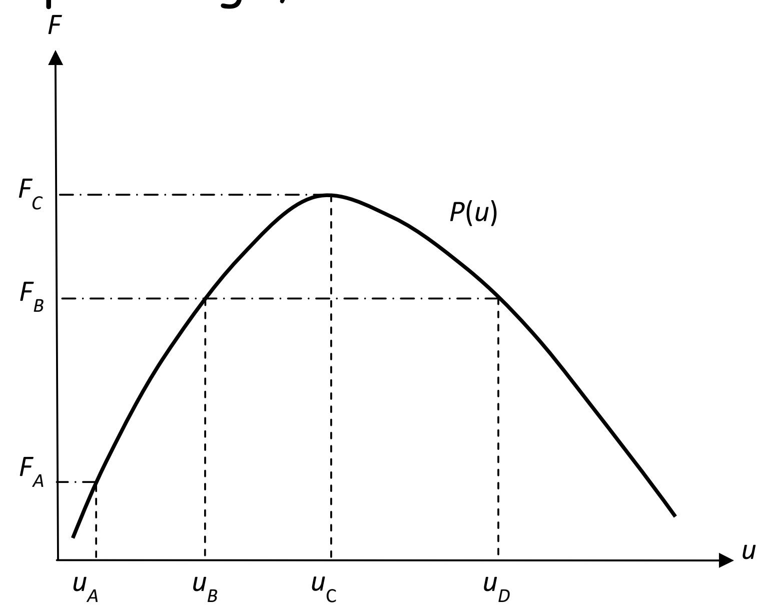
$$F^n = \frac{T^n - T_{start}}{T_{end} - T_{start}} F \qquad\qquad T^n = n \times \Delta T \leq T_{end}$$

- Automatic time stepping
  - Increase/decrease next load increment based on the number of convergence iteration at the current load
  - User provide initial load increment, minimum increment, and maximum increment
  - Bisection of load increment when not converged

### 3.4.6 FORCE CONTROL VS. DISPLACEMENT CONTROL

- Force control: gradually increase the applied forces and find equilibrium configuration
- Displ. control: gradually increase the prescribed displacements
  - Applied load can be calculated as a reaction
  - More stable than force control.
  - Useful for softening, contact, snap-through, etc.



### 3.4.7 NONLINEAR SOLUTION STEPS

1. Initialization: $\mathbf{d}^0 = \mathbf{0};\ i = 0$

2. Residual Calculation $\mathbf{R}^i = \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$

3. Convergence Check (If converged, stop)

4. Linearization

   Calculate tangent stiffness $\mathbf{K}_T^i(\mathbf{d}^i)$

5. Incremental Solution:

   Solve $\mathbf{K}_T^i(\mathbf{d}^i)\Delta\mathbf{d}^i = \mathbf{R}^i$

6. State Determination

   Update displacement and stress
   $$\mathbf{d}^{i+1} = \mathbf{d}^i + \Delta\mathbf{d}^i$$
   $$\sigma^{i+1} = \sigma^i + \Delta\sigma^i$$

7. Go To Step 2

## NONLINEAR SOLUTION STEPS CONT.

- <u>State determination</u>
  - For a given nodal displ $\mathbf{d}^k$, determine current state (strain, stress, etc)

$$\mathbf{u}^k(\mathbf{x}) = \mathbf{N}(\mathbf{x}) \cdot \mathbf{d}^k \qquad \varepsilon^k = \mathbf{B} \cdot \mathbf{d}^k \qquad \sigma^k = f(\varepsilon^k)$$

  - Sometimes, stress cannot be determined using strain alone

- <u>Residual calculation</u>
  - Applied nodal force – Nodal forces due to internal stresses

Weak form:

$$\iiint_\Omega \varepsilon(\bar{\mathbf{u}})^T \sigma \, d\Omega = \iint_{\Gamma_s} \bar{\mathbf{u}}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \bar{\mathbf{u}}^T \mathbf{f}^b \, d\Omega, \quad \forall \bar{\mathbf{u}}$$

Discretization:

$$\bar{\mathbf{d}}^T \left( \iiint_\Omega \mathbf{B}^T \sigma \, d\Omega = \iint_{\Gamma_s} \mathbf{N}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \mathbf{N}^T \mathbf{f}^b \, d\Omega \right), \quad \forall \bar{\mathbf{d}}$$

Residual:

$$\mathbf{R}^k = \iint_{\Gamma_s} \mathbf{N}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \mathbf{N}^T \mathbf{f}^b \, d\Omega - \iiint_\Omega \mathbf{B}^T \sigma^k \, d\Omega$$

**EXAMPLE – LINEAR ELASTIC MATERIAL**

- Governing equation (Scalar equation)

$$\iiint_{\Omega} \varepsilon(\bar{\mathbf{u}})^T \boldsymbol{\sigma}\, d\Omega = \iint_{\Gamma_s} \bar{\mathbf{u}}^T \mathbf{t}\, d\Gamma + \iiint_{\Omega} \bar{\mathbf{u}}^T \mathbf{f}^b\, d\Omega$$

$$\bar{\mathbf{u}} = \mathbf{N} \cdot \bar{\mathbf{d}}$$
$$\varepsilon(\bar{\mathbf{u}}) = \mathbf{B} \cdot \bar{\mathbf{d}}$$

Collect $\bar{\mathbf{d}}$

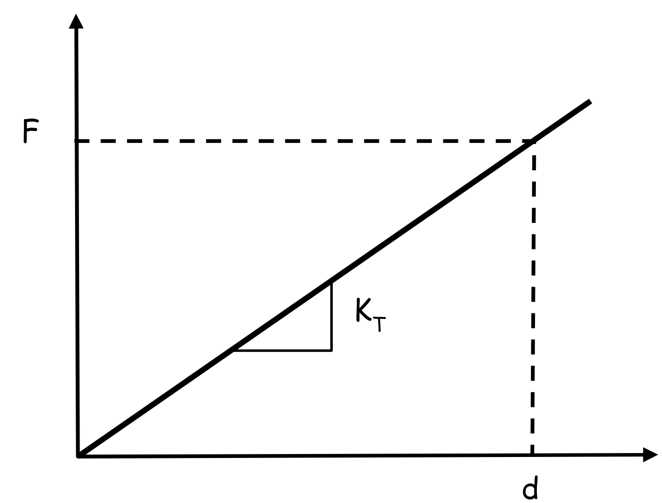
$$\bar{\mathbf{d}}^T \left( \underbrace{\iiint_{\Omega} \mathbf{B}^T \boldsymbol{\sigma}\, d\Omega}_{\mathbf{P}(\mathbf{d})} = \underbrace{\iint_{\Gamma_s} \mathbf{N}^T \mathbf{t}\, d\Gamma + \iiint_{\Omega} \mathbf{N}^T \mathbf{f}^b\, d\Omega}_{\mathbf{F}} \right)$$

- Residual $\mathbf{R} = \mathbf{F} - \mathbf{P}(\mathbf{d})$

- Linear elastic material

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} = \mathbf{C} \cdot \mathbf{B} \cdot \mathbf{d}$$

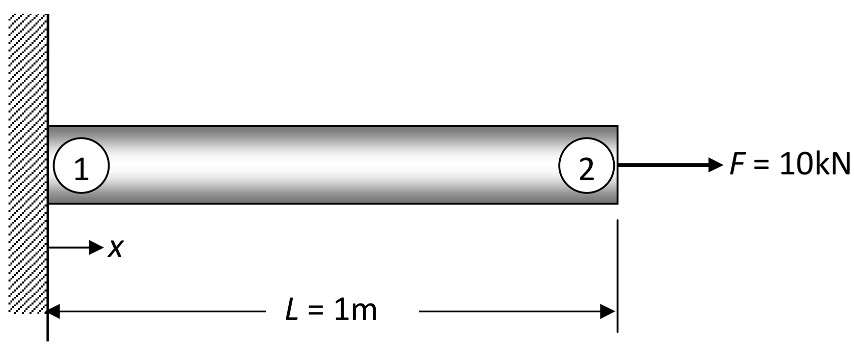
$$\mathbf{K}_T = \frac{\partial \mathbf{P}(\mathbf{d})}{\partial \mathbf{d}} = \iiint_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B}\, d\Omega$$

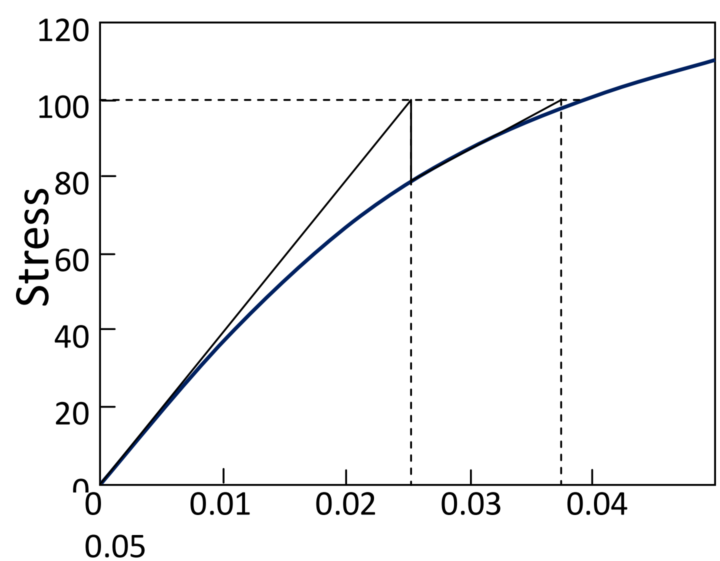


**EXAMPLE – NONLINEAR BAR**

- Rubber bar $\sigma = E\tan^{-1}(m\varepsilon)$
- Discrete weak form $\bar{\mathbf{d}}^T\int_0^L \mathbf{B}^T\sigma A\,dx = \bar{\mathbf{d}}^T\mathbf{F}$
- Scalar equation $R = F - \int_0^L \frac{\sigma A}{L}dx$

$$\Rightarrow R = F - \sigma(d)A$$



## EXAMPLE – NONLINEAR BAR CONT.

- Jacobian

$$\frac{dP}{dd} = \frac{d\sigma(d)}{dd}A = \frac{d\sigma}{d\varepsilon}\frac{d\varepsilon}{dd}A = \frac{1}{L}mAE\cos^2\left(\frac{\sigma}{E}\right)$$

- N-R equation

$$\left[\frac{1}{L}mAE\cos^2\left(\frac{\sigma^k}{E}\right)\right]\Delta d^k = F - \sigma^k A$$

- Iteration 1

$$\frac{mAE}{L}\Delta d^0 = F$$

$$d^1 = d^0 + \Delta d^0 = 0.025m$$
$$\varepsilon^1 = d^1 / L = 0.025$$
$$\sigma^1 = E\tan^{-1}(m\varepsilon^1) = 78.5MPa$$

- Iteration 2

$$\left[\frac{mAE}{L}\cos^2\left(\frac{\sigma^1}{E}\right)\right]\Delta d^1 = F - \sigma^1 A$$

$$d^2 = d^1 + \Delta d^1 = 0.0357m$$
$$\varepsilon^2 = d^2 / L = 0.0357$$
$$\sigma^2 = E\tan^{-1}(m\varepsilon^2) = 96MPa$$

### 3.4.8 N-R OR MODIFIED N-R?

- It is always recommended to use the Incremental Force Method
  - Mild nonlinear: ~10 increments
  - Rough nonlinear: 20 ~ 100 increments
  - For rough nonlinear problems, analysis results depends on increment size
- Within an increment, N-R or modified N-R can be used
  - N-R method calculates $K_T$ at every iteration
  - Modified N-R method calculates $K_T$ once at every increment
  - N-R is better when: mild nonlinear problem, tight convergence criterion
  - Modified N-R is better when: computation is expensive, small increment size, and when N-R does not converge well
- Many FE programs provide <u>automatic stiffness update option</u>
  - Depending on convergence criteria used, material status change, etc

# 3.5 CONVERGENCE CRITERIA

Since the solution of non linear problems is obtained through sequential approximations, it is necessary to introduce a proper tolerance measures to quantify the error in the solution and stop the iterative process.

Several measures of the degree of convergence have been introduced:

- Most analysis programs provide three convergence criteria

- Work, displacement, load (residual)

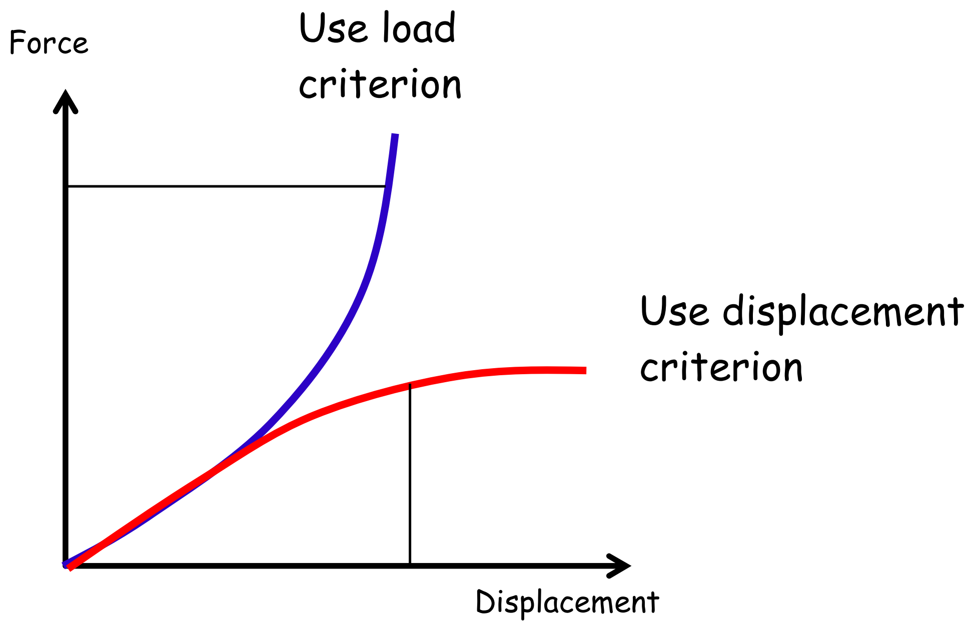
- Work = displacement * load

- At least two criteria needs to satisfy the convergence

- Traditional convergence criterion is load (residual)

  - Equilibrium between internal and external forces $\quad \mathbf{P(d)} = \mathbf{F(d)}$

- Use displacement criterion for load insensitive system



### 3.5.1 DISPLACEMENT CONVERGENCE CRITERION

The displacement error is defined as the ratio between the norm of the displacement increment and the current norm of the displacement vector:

$$e_d = \frac{\left\| d\mathbf{D}^i \right\|_2}{\left\| \mathbf{D}^i \right\|_2} \leq t_d$$

$$\left\| d\mathbf{D}^i \right\|_2 = \sqrt{\sum_{j=1}^{Nnodes} dD_{jx}^2 + dD_{jy}^2 + dD_{jz}^2 + \ldots.}$$

where $t_d$ is the assumed error tolerance for such a criterion.

### 3.5.2 UNBALANCED NODAL FORCES CONVERGENCE CRITERION

The total force error is defined as the ratio between the norm of the unbalanced force difference between two subsequent iterations and the current norm of the unbalanced force vector:

$$e_f = \frac{\left\|\boldsymbol{\psi}^i\right\|_1 - \left\|\boldsymbol{\psi}^{i-1}\right\|_1}{\left\|\boldsymbol{\psi}^i\right\|_1} \leq t_f$$

$$\left\|\boldsymbol{\psi}^i\right\|_1 = \sum_{j=1}^{Nnodes} \left|\Psi_j^i\right|$$

where $t_f$ is the assumed error tolerance for such a criterion.

### 3.5.3 UNBALANCED SINGLE NODAL FORCES CONVERGENCE CRITERION

The single nodal force error is defined as the maximum ratio between the norm of the nodal unbalanced force and the total current norm of the nodal applied force vector:

$$e_{f1} = \max_{j=1,Nnodes} \frac{\left|\mathbf{\Psi}_j^i\right|}{\left|\mathbf{F}_j^i + \mathbf{R}_j^i\right|} \le t_{f1}$$

where $t_{f1}$ is the assumed error tolerance for such a criterion.

### 3.5.4 ENERGY CONVERGENCE CRITERION

The error in term of energy is defined as the ratio between the elastic energy associated with the displacement increment and the current total energy:

$$e_{En} = \frac{1}{2}\frac{d\mathbf{D}^t\mathbf{K}_T d\mathbf{D}}{E_{tot}} \le t_{En}$$

where $t_{En}$ is the assumed error tolerance for such a criterion.

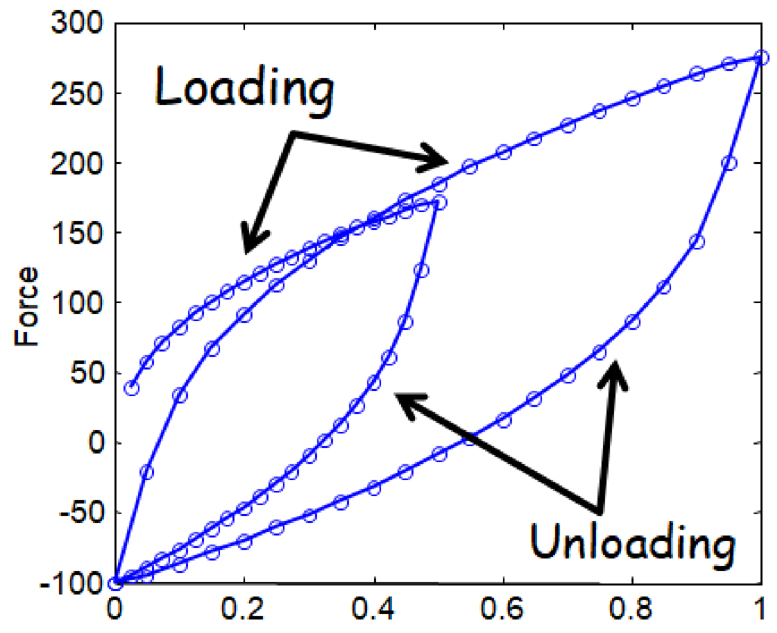
This approach embeds information coming from both the <u>displacements</u> and the <u>forces</u>.

### 3.5.5 ACCURACY VS. CONVERGENCE

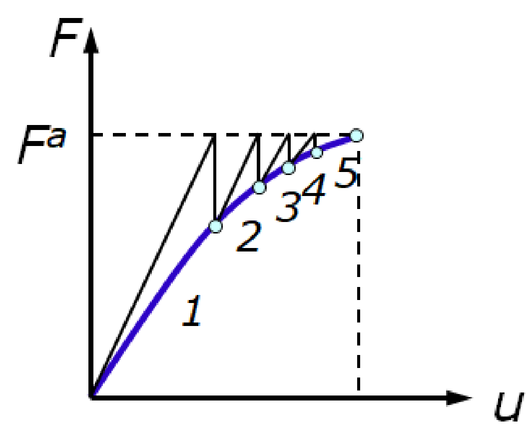
- Nonlinear solution procedure requires:
  - Internal force $\mathbf{P}(\mathbf{d})$
  - Tangent stiffness $\mathbf{K}_T(\mathbf{d}) = \dfrac{\partial \mathbf{P}}{\partial \mathbf{d}}$
  - They are often implemented in the same routine
- Internal force $\mathbf{P}(\mathbf{d})$ needs to be accurate
  - We solve equilibrium of $\mathbf{P}(\mathbf{d}) = \mathbf{F}$
- Tangent stiffness $\mathbf{K}_T(\mathbf{d})$ contributes to convergence
  - Accurate $\mathbf{K}_T(\mathbf{d})$ provides quadratic convergence near the solution
  - Approximate $K_T(d)$ requires more iterations to converge
  - Wrong $\mathbf{K}_T(\mathbf{d})$ causes lack of convergence

### 3.5.6 SOLUTION STRATEGIES

- <u>Load Increment</u> (substeps)
  - Linear analysis concerns max load
  - Nonlinear analysis depends on load path (history)
  - Applied load is gradually increased within a load step
  - Follow load path, improve accuracy, and easy to converge




- <u>Convergence Iteration</u>
  - Within a load increment, an iterative method (e.g., NR method) is used to find nonlinear solution
  - Bisection, linear search, stabilization, etc

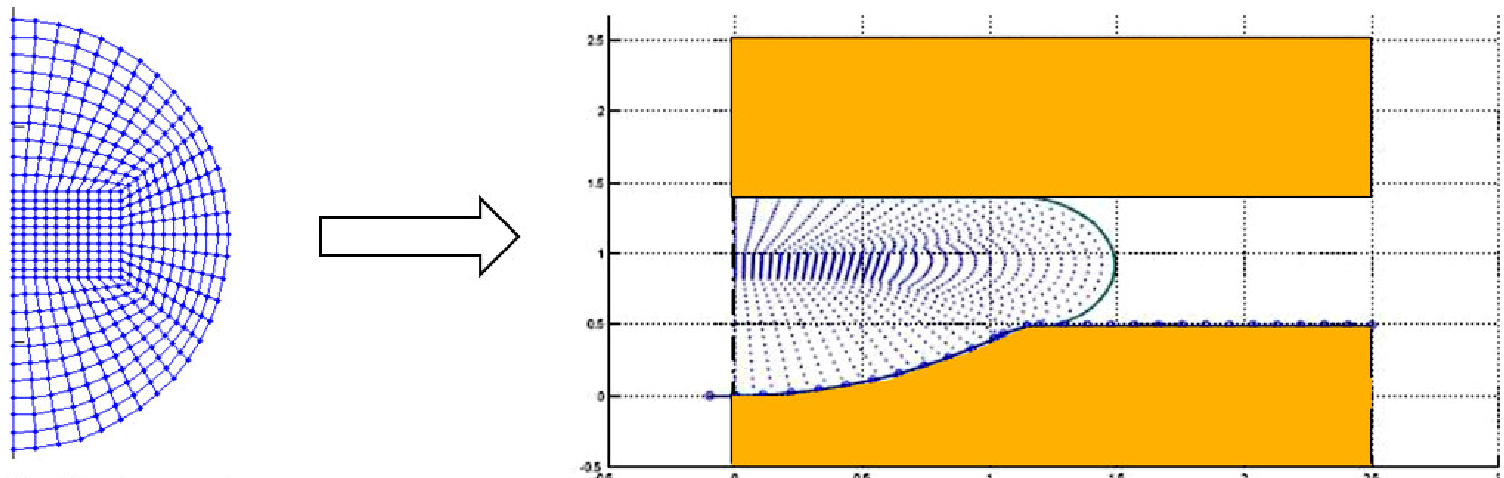


- Automatic (Variable) Load Increment
  - Also called Automatic Time Stepping
  - Load increment may not be uniform
  - When convergence iteration diverges, the load increment is halved
  - If a solution converges in less than 4 iterations, increase time increment by 25%
  - If a solution converges in more than 8 iterations, decrease time increment by 25%

- Subincrement (or bisection)
  - When iterations do not converge at a given increment, analysis goes back to previously converged increment and the load increment is reduced by half
  - This process is repeated until max number of subincrements is reached

- <u>Mesh distortion</u>

- Most FE programs stop analysis when mesh is distorted too much
- Initial good mesh may be distorted during a large deformation
- Many FE programs provide remeshing capability, but it is still inaccurate or inconvenient
- It is best to make mesh in such a way that the mesh quality can be maintained after deformation (need experience)



Initial mesh

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